International Journal of Theoretical Physics, Vol. 42, No. 5, May 2003 (© 2003)

Coexistence and Joint Measurability in Quantum Mechanics[1]

Pekka Lahti[2]

Received April 9, 2003

This talk is a survey of the question of joint measurability of coexistent observables and it is based on the monograph *Operational Quantum Physics* (Busch *et al.*, Springer-Verlag, Berlin, 1997) and on the papers (Lahti *et al.*, *Journal of Mathematical Physics* **39**, 6364–6371, 1998; Lahti and Pulmannova, *Reports on Mathematical Physics* **39**, 339–351, 1997; **47**, 199–212, 2001).

KEY WORDS: coexistence; commensurability; coexistent observables; joint measurability; sequential measurements.

1. INTRODUCTION: COEXISTENT OBSERVABLES

The question on the possibility of measuring together two or more physical quantities lies at the hearth of quantum mechanics. Various notions and formulations have been employed to investigate this issue. von Neumann's (1955) analysis of simultaneous measurability of physical quantities in terms of commutativity of the self-adjoint operators representing those quantities is the starting point of much of the subsequent work. In particular, the investigations of Varadarajan (1962), Gudder (1968), Hardegree (1977), Pulmannová (1980), and Ylinen (1985) constitute an important line of research following von Neumann's approach.

The representation of observables as positive operator measures forces one to go beyond von Neumann's framework. Moreover, the simultaneity of the involved measurements, that is, the fact that the measurements are performed at the same time point, is, perhaps, not the most crucial aspect of this problem. Therefore, in that wider context, the notion of coexistence of observables has been chosen to describe the physical possibility of measuring together two or more quantities. This concept is due to Günther Ludwig (1964) and it was further elaborated e.g., in Ludwig (1967), Hellwig (1969), Neumann (1970), and Kraus (1983). An extensive operational analysis of this notion is presented in Ludwig (1983).

[1] This paper was presented at Quantum Composite Systems 2002, Ustron, Poland, Sept. 3–7, 2002.
[2] Department of Physics, University of Turku, FIN-20014 Turku, Finland; e-mail: pekka.lahti@utu.fi.

Let $\mathcal{H}$ be a complex separable Hilbert space, $\mathcal{L}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, Ω a nonempty set, and $\mathcal{A}$ a sigma algebra of subsets of Ω. We call a positive normalized operator measure $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ an observable and we refer to $(\Omega, \mathcal{A})$ as the value space of E.

Let E, E_1, and E_2 be any three observables with the value spaces $(\Omega, \mathcal{A})$, $(\Omega_1, \mathcal{A}_1)$, and $(\Omega_2, \mathcal{A}_2)$, let ran $(E) = \{E(X) | X \in \mathcal{A}\}$ denote the range of E.

Definition 1.1. Observables $E_1 : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ are coexistent if there is an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that

$$\text{ran}\,(E_1) \cup \text{ran}\,(E_2) \subseteq \text{ran}\,(E),$$

that is, for each $X \in \mathcal{A}_1$, and $Y \in \mathcal{A}_2$, $E_1(X) = E(Z_X)$, and $E_2(Y) = E(Z_Y)$ for some $Z_X, Z_Y \in \mathcal{A}$.

The notion of coexistence of observables is a rather general notion and it seems to be open to characterizations only under further specifications. They will be studied next.

2. FUNCTIONALLY COEXISTENT OBSERVABLES

Definition 2.1. Observables $E_1 : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ are functions of an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ if there are (measurable) functions $f_1 : \Omega \to \Omega_1$ and $f_2 : \Omega \to \Omega_2$ such that for each $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$,

$$E_1(X) = E(f_1^{-1}(X)), \qquad E_2(Y) = E(f_2^{-1}(Y)).$$

In that case we say that E_1 and E_2 are functionally coexistent.

As an immediate observation, one has the following proposition:

Proposition 2.2. *Functionally coexistent observables are coexistent.*

It is an open question whether coexistent observables are functionally coexistent. In what follows we shall investigate conditions under which coexistent observables are functionally coexistent and we shall work out some characterizations for functional coexistence. We start with another simple observation.

Proposition 2.3. *Two-valued observables E_1 and E_2 are coexistent if and only if they are functionally coexistent.*

Proof: To demonstrate this fact, let $\{\omega, \omega'\}$ and $\{\xi, \xi'\}$ be two point value sets of the observables E_1 and E_2, with ran $(E_1) = \{O, A_1, I - A_1, I\}$ and ran

$(E_2) = \{O, A_2, I - A_2, I\}$, respectively, and let E be an observable such that $E(X) = A_1$ and $E(Y) = A_2$. Consider the partition $\mathcal{R} = \{X \cap Y, X' \cap Y, X \cap Y', X' \cap Y'\}$ of the value space Ω of E into disjoint $\mathcal{A}$-sets, and let $1 \mapsto E(X \cap Y), 2 \mapsto E(X' \cap Y), 3 \mapsto E(X \cap Y'), 4 \mapsto E(X' \cap Y')$ constitute a corresponding coarsegrained observable $E^{\mathcal{R}}$ of E. The maps $f_1 : 1, 3 \mapsto \omega; 2, 4 \mapsto \omega'$, and $f_2 : 1, 2 \mapsto \xi; 3, 4 \mapsto \xi'$, allow one to write $A_1 = E^{\mathcal{R}}(f_1^{-1}(\omega)) = E(X \cap Y) + E(X \cap Y')$ and $A_2 = E^{\mathcal{R}}(f_2^{-1}(\xi)) = E(X \cap Y) + E(X' \cap Y)$, showing that the two-valued observables are functionally coexistent. □

Let $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$ denote the product space of the measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, with $\mathcal{A}_1 \times \mathcal{A}_2 = \{(X, Y) | X \in \mathcal{A}_1, Y \in \mathcal{A}_2\}$.

Definition 2.4. A positive operator function $B : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ is a positive operator bimeasure, if for each $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$ the partial functions

$$\mathcal{A}_2 \ni Y \mapsto B(X, Y) \in \mathcal{L}(\mathcal{H}),$$

$$\mathcal{A}_1 \ni X \mapsto B(X, Y) \in \mathcal{L}(\mathcal{H}),$$

are positive operator measures. If $B(\Omega_1, \Omega_2) = I$, we say that B is a biobservable. Observables $E_1 : \mathcal{A}_1 \mapsto \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ have a biobservable if there is a positive operator bimeasure $B : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ such that for all $X \in \mathcal{A}_1, Y \in \mathcal{A}_2$,

$$E_1(X) = B(X, \Omega_2),$$

$$E_2(Y) = B(\Omega_1, Y).$$

To combine observables into new observables, biobservables, or joint observables, to be defined below, some continuity properties are needed. It would suffice to assume that Ω is a Hausdorff space, $\mathcal{A} = \mathcal{B}(\Omega)$ its Borel σ-algebra, and to require that the involde measures are Radon measures of $\mathcal{B}(\Omega)$ (Berg *et al.*, 1984). In physical applications the value spaces of observables are usually, if not always, equipped with locally compact metrisable and separable topologies. In Ludwig (1983) some operational justification for that structure of a value space is also given. The measures on the Borel σ-algebras of such spaces are automatically Radon measures (Halmos, 1950). In particular, this is the case for $(\Omega, \mathcal{B}(\Omega))$ being the real or complex Borel spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, or their n-fold Cartesian products. To avoid the technical assumptions on Radon measures, I assume from nowon that the value spaces of the observables are *locally compact metrisable and separable topological spaces* and, for short, I call them simply *Borel spaces*. Where this assumption is superfluous, I go on to use the notation $(\Omega, \mathcal{A})$ to emphasize that no topological assumptions are needed.

Let $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$ be two Borel spaces, and let $\mathcal{B}(\Omega_1 \times \Omega_2)$ denote the Borel σ-algebra of $\Omega_1 \times \Omega_2$.

Definition 2.5. Observables $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ have a joint observable if there is an observable $F : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{H})$ such that for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$,

$$E_1(X) = F(X \times \Omega_2), \tag{1}$$

$$E_2(Y) = F(\Omega_1 \times Y). \tag{2}$$

Theorem 2.6. *Let $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$ be two Borel spaces. For any two observables $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ the following three conditions are equivalent:*

(i) *E_1 and E_2 have a biobservable;*
(ii) *E_1 and E_2 have a joint observable;*
(iii) *E_1 and E_2 are functionally coexistent.*

Proof: (i) $\Rightarrow$ (ii). Let B be a biobservable associated with E_1 and E_2. Then, for any $\varphi \in \mathcal{H}$, the bimeasure $X \times Y \mapsto \langle \varphi | B(X, Y)\varphi \rangle$ determines a unique measure $\mu(B, \varphi)$ on $(\Omega_1 \times \Omega_2, \mathcal{B}(\Omega_1 \times \Omega_2))$ such that for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$,

$$\mu(B, \varphi)(X \times Y) = \langle \varphi | B(X, Y)\varphi \rangle$$

(see Theorem 1.10 of Berg *et al.*, 1984, p. 24). Putting $F_{\varphi,\varphi}(Z) = \mu(B, \varphi)(Z)$ for all $\varphi \in \mathcal{H}$, $Z \in \mathcal{B}(\Omega_1 \times \Omega_2)$, one defines through the polarization identity and the Frèchet–Riesz theorem an observable $F : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{H})$ with the property

$$F(X \times \Omega_2) = B(X, \Omega_2) = E_1(X),$$

$$F(\Omega_1 \times Y) = B(\Omega_1, Y) = E_2(Y),$$

for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$. (ii) $\Rightarrow$ (iii). Let now F be a joint observable of E_1 and E_2, and let π_1 and π_2 be the respective coordinate projections $\Omega_1 \times \Omega_2 \to \Omega_1$, $\Omega_1 \times \Omega_2 \to \Omega_2$. Then $E_1(X) = F(\pi_1^{-1}(X))$ and $E_2(Y) = F(\pi_2^{-1}(Y))$, showing that E_1 and E_2 are functionally coexistent. (iii) $\Rightarrow$ (i). If $E_1(X) = E(f_1^{-1}(X))$ and $E_2(Y) = E(f_2^{-1}(Y))$, for some observable $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$, and some Borel functions $f_i : \Omega \to \Omega_i$, $i = 1, 2$, then $\langle \varphi | B(X, Y)\varphi \rangle := E_{\varphi,\varphi}(f_1^{-1}(X) \cap f_2^{-1}(Y))$, $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$, $\varphi \in \mathcal{H}$, defines a biobservable B with the desired properties. □

Example 2.7. Assume that the observable E_1 and E_2 are mutually commuting, that is, $E_1(X)E_2(Y) = E_2(Y)E_1(X)$ for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$. The map

$$(X, Y) \mapsto E(X, Y) := E_1(X)E_2(Y),$$

is then a biobservable. Indeed, $E(\Omega_1, \Omega_2) = I$, whereas the positivity of $E(X, Y)$ follows from the commutativity and positivity of $E_1(X)$ and $E_2(Y)$. The measure properties of E_1 and E_2 and the (weak) continuity of the operator product imply that the partial functions $X \mapsto E(X, Y)$, $Y \in \mathcal{B}(\Omega_2)$, and $Y \mapsto E(X, Y)$, $X \in \mathcal{B}(\Omega_1)$, are positive operator measures. Theorem 2.6 thus implies that any two mutually commuting observables have a joint observable and they are functionally coexistent. The mutual commutativity of E_1 and E_2 is, however, not necessary for any of the conditions of that theorem, as will become evident in subsequent discussion.

Remark 2.8. There is an alternative formulation of functional coexistence of observables, which actually goes back to Ludwig (1983, D.3.1, p. 153). Indeed, one could say that observables E_1 and E_2 are functionally coexistent if there is an observables E and σ-homomorphisms $h_1 : \mathcal{B}(\Omega_1) \to \mathcal{B}(\Omega)$ and $h_2 : \mathcal{B}(\Omega_2) \to \mathcal{B}(\Omega)$ such that $E_1(X) = E(h_1(X))$ and $E_2(Y) = E(h_2(Y))$ for every X and Y. If this is the case, then the map $(X, Y) \mapsto E(h_1(X) \cap h_2(Y))$ is a bimeasure, and thus E_1 and E_2 are functionally coexistent also in the sense of Definition 2.1.

3. REGULARLY COEXISTENT OBSERVABLES

In a realist interpretation of quantum mechanics the notion of regular effect is an important one: A nontrivial effect B is regular if its spectrum extends both below as well as above the value $\frac{1}{2}$. For a further analysis of this notion the reader may consult (Busch *et al.*, 1997). Its relevance here follows from that fact that regular observables are characterized by their Boolean range.

Definition 3.7. An observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is called regular, if for any $X \in \mathcal{A}$, such that $O \neq E(X) \neq I$,

$$E(X) \not\leq \frac{1}{2}I, \qquad \frac{1}{2}I \not\leq E(X).$$

Clearly, an observable E is regular if and only if for any $O \neq E(X) \neq I$ neither $E(X) \leq E(X)'$ nor $E(X)' \leq E(X)$.

Lemma 3.2. *Let $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a regular observable. If $(A_i)_{i\in\mathbb{N}} \subset$ ran (E) is a summable sequence, that is, $A_1 + \cdots + A_n \leq I$ for each $n \in \mathbb{N}$, then* $\text{sub}_{p_{n\in\mathbb{N}}}\{A_1 + \cdots + A_n\} \in \text{ran}\,(E)$. *Moreover,* ran *(E) is a Boolean algebra (with respect to the order and complement inhereted from the set of effects $\mathcal{E}(\mathcal{H})$), and E is a Boolean σ-homomorphism $A \to$* ran (E).

Proof: The proof follows that of Lahti and Pulmannova (2001, Theorem 4.1). Let $A_1, A_2 \in \text{ran}\,(E)$ be such that $A_1 + A_2 \leq I$ and assume that $A_1 = E(X)$,

$A_2 = E(Y)$. Then $X = X \cap Y \cup (X \setminus (X \cap Y))$, $Y = X \cap Y \cup (Y \setminus (X \cap Y))$. Hence $E(X \cap Y) \leq E(X) = A_1$, $E(X \cap Y) \leq E(Y) = A_2$. Since $A_2 \leq I - A_1$, it follows that $E(X \cap Y) \leq E(X \cap Y)'$, so that by the regularity assumption, $E(X \cap Y) = O$. Therefore $A_1 = E(X_1)$ and $A_2 = E(Y_1)$, where $X_1 := X \setminus (X \cap Y)$, $Y_1 := Y \setminus (X \cap Y)$ are disjoint sets. So we get $A_1 + A_2 = E(X_1) + E(Y_1) = E(X_1 \cup Y_1) \in$ ran (E). We note also that $X \cap Y_1 = \emptyset$ and $A_1 + A_2 = E(X \cup Y_1)$. This observation will be used in the next paragraph.

Assume next that $(A_i)_{i \in \mathbb{N}}$ is a summable sequence in ran (E). Using the above argument, we find disjoint sets X_1, X_2 such that $A_1 = E(X_1)$, $A_2 = E(X_2)$. Now we proceed by induction. Assume that we have already found disjoint sets $X_1, \ldots, X_{n-1}$ such that $A_i = E(X_i)$, $i = 1, \ldots, n-1$. Then $A_1 + \cdots + A_{n-1} = E(X_1 \cup X_2 \cup \ldots \cup X_{n-1})$. By the summability assumption $(A_1 + \cdots + A_{n-1}) \leq I - A_n$. Therefore, there is a set $X_n \in \mathcal{A}$ such that $(X_1 \cup \ldots \cup X_{n-1}) \cap X_n = \emptyset$, and $A_n = E(X_n)$. Thus we find a sequence X_i, $i \in \mathbb{N}$, of disjoint sets such that $A_i = E(X_i)$, $i \in \mathbb{N}$. From the σ-additivity of E we obtain $E(\cup_i X_i) = \sum_i E(X_i) = \sum_i A_i$, which shows that ran (E) is closed under sums of summable sequences.

Let $E(X), E(Y) \in$ ran (E). We will prove that

$$E(X \cap Y) = E(X) \wedge_{\mathrm{ran}(E)} E(Y),$$

that is, $E : \mathcal{A} \to$ ran (E) is a $\wedge$-morphism. Evidently, $E(X \cap Y) \leq E(X), E(Y)$. Assume that for some $Z \in \mathcal{A}$, $E(Z) \leq E(X), E(Y)$. We can write $Z = (Z \cap X \cap Y) \cup (Z \cap (X \cap Y)')$. Moreover,

$$\begin{aligned} E(Z \cap (X \cap Y)') &= E(Z \cap (X' \cup Y')) \\ &= E((Z \cap X' \cap Y) \cup (Z \cap X' \cap Y') \cup (Z \cap X \cap Y')) \\ &= E(Z \cap X' \cap Y) + E((Z \cap X' \cap Y') + E((Z \cap X \cap Y')) \\ &\leq E(Z) \leq E(X), E(Y). \end{aligned}$$

But we also have $E(Z \cap X' \cap Y) \leq E(X')$, $E(Z \cap X \cap Y') \leq E(Y')$, $E(Z \cap X' \cap Y') \leq E(X'), E(Y')$, so that the effects $E(Z \cap X' \cap Y)$, $E(X \cap X' \cap Y')$, and $E(Z \cap X \cap Y')$ are irregular and thus equal O. Therefore also $E(Z \cap (X \cap Y)') = O$. Thus $E(Z) = E(Z \cap X \cap Y) \leq E(X \cap Y)$. This concludes the proof that $E(X \cap Y) = E(X) \wedge_{\mathrm{ran}(E)} E(Y)$. By de Morgan laws one gets the dual result: for any $X, Y \in \mathcal{A}$, $E(X \cup Y) = E(X) \vee_{\mathrm{ran}(E)} E(Y)$. Moreover, if the sets X and Y are disjoint, then $E(X) \vee_{\mathrm{ran}(E)} E(Y) = E(X) + E(Y)$. Also, if $(X_i) \subset \mathcal{A}$ is a disjoint sequence, then

$$E\left(\cup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} E(X_i) = \bigvee_{\mathrm{ran}(E)} \{E(X_i) | i \in \mathbb{N}\}.$$

To prove that ran (E) is a Boolean algebra, it remains to prove distributivity. This follows immediately from the fact that E is a $\wedge$-morphism and a $\vee$-morphism from a Boolean set. $\square$

Corollary 3.1. *The range ran (E) of an observable E is a Boolean algebra (with the ordering inherited from $\mathcal{E}(\mathcal{H})$) if and only if E is regular.*

Proof: We have to prove the "only if" part. Hence, assume that ran (E) is Boolean, and let $E(X)$ be an irregular element. Then $E(X) \leq E(X)'$, which in a Boolean algebra implies that $E(X) = 0$. $\square$

Theorem 3.3. *For any two observables $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$, if there is a regular observable $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ such that* ran $(E_1) \cup$ ran $(E_2) \subseteq$ ran (E), *then E_1 and E_2 are functionally coexistent.*

Proof: If E is regular, then from ran $(E_1) \cup$ ran $(E_2) \subseteq$ ran (E) it follows that also E_1 and E_2 are regular. Therefore, by Lemma 3.2, all the ranges ran (E_1), ran (E_2), ran (E) are Boolean. From this and from the fact that ran $(E_1) \cup$ ran $(E_2) \subseteq$ ran (E) it then follows that the map $(X, Y) \mapsto E_1(X) \wedge_{\mathrm{ran}(E)} E_2(Y)$ is a biobservable of E_1 and E_2. Indeed, for a fixed $Y \in \mathcal{B}(\Omega_2)$, if $(X_i) \subset \mathcal{B}(\Omega_1)$ is a disjoint sequence, then

$$
\begin{aligned}
E_1(\cup X_i) \wedge_{\mathrm{ran}(E)} E_2(Y) &= \left(\sum E_1(X_i)\right) \wedge_{\mathrm{ran}(E)} E_2(Y) \\
&= \left(\sum E(Z_{X_i})\right) \wedge_{\mathrm{ran}(E)} E_2(Y) \\
&= \left(\bigvee_{\mathrm{ran}(E)} E(Z_{X_i})\right) \wedge_{\mathrm{ran}(E)} E_2(Y) \\
&= \bigvee_{\mathrm{ran}(E)} (E(X_i) \wedge_{\mathrm{ran}(E)} E_2(Y)),
\end{aligned}
$$

where $(Z_{X_i}) \subset \mathcal{B}(\Omega)$ is a disjoint sequence such that $E(Z_{X_i}) = E_1(X_i)$ (which exists since $(E(X_i)) \subset$ ran (E) is summable). Similarly, one shows that for a fixed $X \in \mathcal{B}(\Omega_1)$, if $(Y_i) \subset \mathcal{B}(\Omega_2)$ is a disjoint sequence, then

$$
E_1(X) \wedge_{\mathrm{ran}(E)} E_2(\cup Y_i) = \bigvee_{\mathrm{ran}(E)} (E(X) \wedge_{\mathrm{ran}(E)} E_2(Y_i)).
$$

Theorem 2.6 now assures that E_1 and E_2 are functionally coexistent. $\square$

In the context of the above theorem we say that observables E_1 and E_2 are *regularly coexistent*. We may then say that regularly coexistent observables are functionally coexistent.

4. PROJECTION AS A VALUE OF AN OBSERVABLE

Projection-valued observables are known to have very special properties. For the coexistence of two observables the fact that one of them is projection-valued implies great simplifications. I start with quoting a well-known result.

Lemma 4.1. *For any positive operator measure $E : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$, if $E(X)^2 = E(X)$ for some $X \in \mathcal{A}$, then $E(X)E(Y) = E(Y)E(X)$ for all $Y \in \mathcal{A}$.*

Proof: Assume that $E(X)^2 = E(X)$ for some $X \in \mathcal{A}$. For any $Y \in \mathcal{A}$, $X \cap Y \subseteq Y$, so that $E(Y) = E(Y \setminus (X \cap Y)) + E(X \cap Y)$, and $E(X) + E(Y) - E(X \cap Y) = E(X) + E(Y \setminus (X \cap Y)) = E(X \cup Y) \leq I$. Therefore, the effects $E(X \cap Y)$ and $E(Y \setminus (X \cap Y))$ are below the projections $E(X)$ and $I - E(X)$, respectively, so that

$$E(X \cap Y) = E(X)E(X \cap Y)E(X),$$

$$E(Y \setminus (Y \cap X)) = (I - E(X))E(Y \setminus (Y \cap X))(I - E(X).$$

Therefore, $E(Y) = E(Y \setminus (X \cap Y)) + E(X \cap Y) = E(X)E(X \cap Y)E(X) + (I - E(X))E(Y \setminus (Y \cap X))(I - E(X)$, which gives through multiplication by $E(X)$ that $E(X)E(Y) = E(Y)E(X)$. □

Corollary 4.2. *Assume that $E_1 : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ are coexistent observables. If one of them is projection-valued, then they are mutually commuting and hence functionally coexistent.*

Proof: Assume that E_1 is projection-valued. Since E_1 and E_2 are coexistent, Lemma 4.1 implies that E_1 and E_2 are commuting: $E_1(X)E_2(Y) = E_2(Y)E_1(X)$ for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$. Thus the map $\mathcal{A}_1 \times \mathcal{A}_2 \ni (X, Y) \mapsto E_1(X)E_2(Y) \in \mathcal{L}(\mathcal{H})$ determines a biobservable of E_1 and E_2, so that, by Theorem 2.6, observables E_1 and E_2 are functionally coexistent. □

5. COMMENSURABILITY

For projection-valued observables the following notion of commensurability, or compatibility, is a further specification of the notion of coexistence. These notions were widely used in the so-called quantum logic approaches to quantum mechanics (see, for instance, Beltrametti and Cassinelli, 1981; Mittelstaedt, 1978; Varadarajan, 1985).

Definition 5.1. Projection-valued observables $E_1 : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ are commensurable, if there is a projection-valued observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that $\text{ran}\,(E_1) \cup \text{ran}\,(E_2) \subseteq \text{ran}\,(E)$.

Theorem 5.2. *Any two projection-valued observables* $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ *and* $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ *are coexistent if and only if they are commensurable.*

Proof: Assume that E_1 and E_2 are coexistent projection-valued observables. By Lemma 4.1 they are mutally commuting. Therefore, $(X, Y) \mapsto E_1(X)E_2(Y)$ is a projection operator bimeasure, so that there is a joint projection-valued observable $E : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{H})$ such that $E(X \times \Omega_2) = E_1(X)$ and $E(\Omega_1 \times Y) = E_2(Y)$. Thus E_1 and E_2 are commensurable. By definition, commensurable observables are coexistent. □

Any two coexistent projection-valued observables E_1 and E_2 are mutually commuting:

$$E_1(X)E_2(Y) = E_2(Y)E_1(X) \quad \text{for all } X \in \mathcal{A}_1, Y \in \mathcal{A}_2. \tag{3}$$

The pioneering result of von Neumann (1955) on commuting self-adjoint operators gives that any two mutually commuting projection-valued observables are (Borel) functions of a third projection-valued observable. We collect these results in the following corollary.

Corollary 5.3. *Let* $(\Omega_1, \mathcal{B}(\Omega_1))$ *and* $(\Omega_2, \mathcal{B}(\Omega_2))$ *be two Borel spaces. For any two projection-valued observables* $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ *and* $E_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ *the following six conditions are equivalent:*

(i) E_1 *and* E_2 *commute;*
(ii) E_1 *and* E_2 *are commensurable;*
(iii) E_1 *and* E_2 *are coexistent;*
(iv) E_1 *and* E_2 *are functionally coexistent;*
(v) E_1 *and* E_2 *have a biobservable;*
(vi) E_1 *and* E_2 *have a joint observable.*

For projection-valued observables E_1 and E_2, their commutativity, or coexistence, or any of the above equivalent formulations, has a natural generalization to a partial commutativity, or partial coexistence. I shall review this question next, the basic results are due to (Hardegree, 1977; Pulmannová, 1980; Ylinen, 1985).

Definition 5.4. For any two projection-valued observables E_1 and E_2, their commutativity domain com (E_1, E_2) consists of those vectors $\varphi \in \mathcal{H}$ for which

$$E_1(X)E_2(Y)\varphi = E_2(Y)E_1(X)\varphi \tag{4}$$

for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$. We say that E_1 and E_2 are commutative if com $(E_1, E_2) = \mathcal{H}$ and totally noncommutative if com $(E_1, E_2) = \{0\}$.

Lemma 5.5. *For any two projection-valued observables E_1 and E_2 their commutativity domain com (E_1, E_2) is a closed subspace of $\mathcal{H}$ and it reduces E_1 and E_2, that is, for any $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$,*

$$E_1(X)(\mathrm{com}\,(E_1, E_2)) \subseteq \mathrm{com}\,(E_1, E_2),$$
$$E_2(Y)(\mathrm{com}\,(E_1, E_2)) \subseteq \mathrm{com}\,(E_1, E_2).$$

Proof: The first claim follows since com (E_1, E_2) can be expressed as the intersection of closed subspaces,

$$\mathrm{com}\,(E_1, E_2) = \cap_{X,Y}\{\varphi \in \mathcal{H} | (E_1(X)E_2(Y) - E_2(Y)E_1(X))\varphi = 0\}.$$

Let $\varphi \in \mathrm{com}\,(E_1, E_2)$. Then for any $Z \in \mathcal{A}_1$, $E_1(Z)\varphi \in \mathrm{com}\,(E_1, E_2)$, since, for all $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$,

$$\begin{aligned} E_2(Y)E_1(X)E_1(Z)\varphi &= E_2(Y)E_1(X \cap Z)\varphi = E_1(X \cap Z)E_2(Y)\varphi \\ &= E_1(X)E_1(Z)E_2(Y)\varphi = E_1(X)E_2(Y)E_1(Z)\varphi. \end{aligned}$$

Similarly, one gets $E_2(Y)$ (com $(E_1, E_2)) \subseteq$ com (E_1, E_2) for each $Y \in \mathcal{A}_2$. □

Theorem 5.6. *Consider two projection-valued observables E_1 and E_2 defined on the Borel spaces $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$, respectively. For any unit vector $\varphi \in \mathcal{H}$, the following conditions are equivalent:*

(i) *$\varphi \in$ com (E_1, E_2),*
(ii) *there is a probability measure $\mu : \mathcal{B}(\Omega_1 \times \Omega_2) \to [0, 1]$ such that*

$$\mu(X \times Y) = \langle\varphi|E_1(X)E_2(Y)\varphi\rangle = \langle\varphi|E_1(X) \wedge E_2(Y)\varphi\rangle$$

for all $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$.

Proof: The restrictions $\widetilde{E_1}$ and $\widetilde{E_1}$ of E_1 and E_2 on com (E_1, E_2) are mutually commuting spectral measures, so that, by Corollary 5.3, the map $X \times Y \mapsto \widetilde{E_1}(X)\widetilde{E_2}(Y) = \widetilde{E_1}(X) \wedge \widetilde{E_2}(Y)$ extends to a joint projection-valued observable $\widetilde{F} : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathrm{com}\,(E_1, E_2))$. But then, for any $\varphi \in \mathrm{com}\,(E_1, E_2)$, and $X \in \mathcal{B}(\Omega_1)$, $Y \in \mathcal{B}(\Omega_2)$, $\widetilde{F}_{\varphi,\varphi}(X \times Y) = \langle\varphi|\widetilde{E_1}(X)\widetilde{E_2}(Y)\varphi\rangle = \langle\varphi E_1(X)E_2(Y)\varphi\rangle$, which concludes the proof. □

Remark 5.7. Let A and B be any two self-adjoint operators in $\mathcal{H}$. According to the spectral theorem for self-adjoint operators, there are unique spectral measures E^A and E^B, defined on the real Borel spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and taking values in $\mathcal{L}(\mathcal{H})$ such that A and B are their respective first moment operators. By definition, A and B commute if and only if all their spectral projection $E^A(X)$ and $E^B(Y)$, $X, Y \in \mathcal{B}(\mathbb{R})$), commute. By a well-known theorem of von Neumann (1955), this is the case exactly when there is a self-adjoint operator C and real Borel function f and g such that $A = f(C)$, $B = g(C)$, that is, $E^A(X) = E^C(f^{-1}(X))$ and $E^B(Y) = E^C(g^{-1}(Y))$ for all $X, Y \in \mathcal{B}(\mathbb{R})$). We recall further that if A and B are bounded self-adjoint operators, then their commutativity is equivalent with the fact that $AB = BA$.

6. SEQUENTIAL MEASUREMENTS

Let $\mathcal{T}(\mathcal{H})$ denote the set of trace class operators on $\mathcal{H}$, and let $\mathcal{S}(\mathcal{H})$ denote its subset of positive trace one operators, the states of the quantum system associated with $\mathcal{H}$. Let $\mathcal{L}(\mathcal{T}(\mathcal{H}))$ denote the set of (trace norm) bounded linear operators on $\mathcal{T}(\mathcal{H})$, which is a complex Banach space with respect to the trace norm. Let $(\Omega, \mathcal{A})$ be a measurable space. A function $\mathcal{I} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ is an *instrument* if for all $T \in \mathcal{S}(\mathcal{H})$ the function

$$\mathcal{A} \ni X \mapsto \mathrm{tr}[\mathcal{I}(X)(T)] \in \mathbb{C}$$

is a probability measure. It follows that the function $X \mapsto E(X)$, defined through

$$\mathrm{tr}[TE(X)] := \mathrm{tr}[\mathcal{I}(X)(T)], \qquad X \in \mathcal{A}, \quad T \in \mathcal{S}(\mathcal{H}),$$

is an observable $\mathcal{A} \to \mathcal{L}(\mathcal{H})$, the *associate observable* of $\mathcal{I}$. It is another matter of fact that each observable E is the associate observable of some instruments $\mathcal{I}$; such instruments are called E-compatible.

Consider any two instruments $\mathcal{I}_1 : \mathcal{A}_1 \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ and $\mathcal{I}_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$, and let E_1 and E_2 be their associate observables. For each $T \in \mathcal{S}(\mathcal{H})$ the function

$$\mathcal{A}_1 \times \mathcal{A}_2 \ni (X, Y) \mapsto \mu_T(X, Y) := \mathrm{tr}[\mathcal{I}_1(X)(\mathcal{I}_2(Y)(T))] \in [0, 1]$$

is a probability bimeasure. By the duality $\mathcal{T}(\mathcal{H})^* \cong \mathcal{L}(\mathcal{H})$, the bimeasures μ_T, $T \in \mathcal{T}(\mathcal{H})$, define a positive operator bimeasure $B : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ such that

$$\mathrm{tr}[TB(X, Y)] = \mu_T(X, Y),$$

for all $T \in \mathcal{S}(\mathcal{H})$, $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$. The partial positive operator measures E_{Ω_1} and E^{Ω_2}, associated with Ω_1 and Ω_2, respectively, are easily seen to be the observables

$$E_{\Omega_1}(Y) := B(\Omega_1, Y) = E_2(Y), \quad Y \in \mathcal{A}_2, \tag{5}$$

$$E^{\Omega_2}(X) := B(X, \Omega_2) = \mathcal{I}_2(\Omega_2)^*(E_1(X)), \quad X \in \mathcal{A})1, \tag{6}$$

where we have used the dual transformation $\mathcal{I}_2(\Omega_2)^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ of the state transformation $\mathcal{I}_2(\Omega_2) : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$; for all $T \in \mathcal{S}(\mathcal{H})$, $A \in \mathcal{L}(\mathcal{H})$

$$\mathrm{tr}[T\mathcal{I}_2(\Omega_2)^*(A)] := \mathrm{tr}[\mathcal{I}_2(\Omega_2)(T)A].$$

We recall that using the dual transformer $\mathcal{I}^* : X \mapsto \mathcal{I}(X)^*$, $X \in \mathcal{A}_2$, the associate observable E of $\mathcal{I}$ can be expressed as $E(X) = \mathcal{I}(X)^*(I)$, $X \in \mathcal{A}$ (see, for instance, Davies (1976)).

The above construction of biobservables shows that any two instruments $\mathcal{I}_1$ and $\mathcal{I}_2$, defined on the Borel spaces $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$, respectively, give rise to a pair of observables for which any of the conditions of Theorem 2.6 is satisfied. These observables depend on the order in which the instruments are applied:

$$\mathrm{tr}[TB_{21}(X, Y)] := \mathrm{tr}[\mathcal{I}_1(X)(\mathcal{I}_2(Y)(T))], \tag{7}$$

$$\mathrm{tr}[TB_{12}(X, Y)] := \mathrm{tr}[\mathcal{I}_2(Y)(\mathcal{I}_1(X)(T))]. \tag{8}$$

In the first case these observables are those given in (5) and (6), in the second case they are given by

$$B_{12}(\Omega_1, Y) = \mathcal{I}_1(\Omega_1)^*(E_2(Y)), \quad Y \in \mathcal{B}(\Omega_2), \tag{9}$$

$$B_{12}(X, \Omega_2) = E_1(X), \quad X \in \mathcal{B}(\Omega_2). \tag{10}$$

Usually, the sequential biobservables B_{21} and B_{12} are different. However, it may happen that they are the same, that is, $B_{12} = B_{21}$. In such a case the observables E_1 and E_2 are, by Theorem 2.6, functionally coexistent.

7. JOINT MEASURABILITY

A measurement scheme for a quantum system associated with a Hilbert space $\mathcal{H}$ is a 4-tuple $\mathcal{M} := \langle \mathcal{K}, W, P, V \rangle$ consisting of a (complex separable) Hilbert space $\mathcal{K}$ (describing the measuring apparatus), a state $W \in \mathcal{S}(\mathcal{K})$ (the initial state of the apparatus), an observable $P : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ (the pointer observable), and a state transformation $V : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{K})$ (a positive trace preserving map which models the measurement coupling). A measurement scheme $\mathcal{M}$ determines an observable $E^{\mathcal{M}} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ through the relation:

$$\mathrm{tr}[TE^{\mathcal{M}}(X)] = \mathrm{tr}[V(T \otimes W)I \otimes P(X)], \qquad T \in \mathcal{S}(\mathcal{H}), \quad X \in \mathcal{A}.$$

This observable is the observable measured by the scheme $\mathcal{M}$. It is a basic result of the quantum theory of measurement that for each observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ there is a measurement scheme $\mathcal{M}$ such that $E = E^{\mathcal{M}}$ (Ozawa, 1984). A measurement scheme $\mathcal{M}$ also determines an instrument $\mathcal{I}^{\mathcal{M}}$:

$$\mathcal{I}^{\mathcal{M}}(T) := \mathrm{tr}_{\mathcal{K}}(V(T \otimes W)I \otimes P(X)), \qquad T \in \mathcal{S}(\mathcal{H}), \quad X \in \mathcal{A},$$

where $\mathrm{tr}_{\mathcal{K}} : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H})$ is the partial trace over the apparatus Hilbert space $\mathcal{K}$. Clearly, $E^{\mathcal{M}}$ is the associate observable of $\mathcal{I}^{\mathcal{M}}$.

Consider now any two observables E_1 and E_2 (of the system with the Hilbert space $\mathcal{H}$). We say that E_1 and E_2 can be *measured together* if there is a measurement scheme $\mathcal{M}$ and two Borel functions (pointer functions) f_1 and f_2 such that

$$E_1(X) = E^{\mathcal{M}}(f_1^{-1}(X)), \qquad X \in \mathcal{A}_1,$$

$$E_2(Y) = E^{\mathcal{M}}(f_2^{-1}(Y)), \qquad Y \in \mathcal{A}_2.$$

It is an immediate observation that the observables E_1 and E_2 can be measured together in the above sense if and only if they are functionally coexistent.

Consider next any two measurement schemes $\mathcal{M}_1$ and $\mathcal{M}_2$. They can be applied sequentially, in either order: first $\mathcal{M}_1$ and then $\mathcal{M}_2$, or first $\mathcal{M}_2$ and then $\mathcal{M}_1$. This corresponds to the application of the instruments $\mathcal{I}^{\mathcal{M}_1}$ and $\mathcal{I}^{\mathcal{M}_2}$ one after the other, in either order. The resulting sequential biobservables B_{21} and B_{12} are obtained from Eqs. (7) and (8). In general, the result of such a sequential measurement depends on the order in which the two measurements are performed. It may happen, however, that the measurements in question are commutative in the sense that their sequential application is order-independent. The observables $E^{\mathcal{M}_1}$ and $E^{\mathcal{M}_2}$ determined by such measurement schemes are functionally coexistent. As is well known, it is highly exceptional that two measurement schemes $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative in this sense.

REFERENCES

Beltrametti, E. and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, MA.

Berg, C., Christensen, J., and Ressel, P. (1984). *Harmonic Analysis on Semigroups*, Springer-Verlag, Berlin.

Busch, P., Grabowski, M., and Lahti, P. (1997). *Operational Quantum Physics*, LNP m31, Springer, Berlin. (2nd corrected printing 1997)

Davies, E. B. (1976). *Quantum Theory of Open System*, Academic Press, London.

Gudder, S. P. (1968). Joint distributions of observables, *Journal of Mathematics and Mechanics* **18**, 325–335.

Halmos, P. (1988). *Measure Theory*, Springer-Verlag, Berlin. (Originally published in 1950)

Hardegree, G. M. (1977). Relative compatibility in conventional quantum mechanics, *Foundations of Physics* **7**, 495–510.

Hellwig, K.-E. (1969). Coexistent effects in quantum mechanics, *International Journal of Theoretical Physics* **2**, 147–155.

Kraus, K. (1983). *States, Effects, and Operations*, Springer-Verlag, Berlin.

Lahti, P. and Pulmannová, S. (1997). Coexistent observables and effects in quantum mechanics. *Reports on Mathematical Physics* **39**, 339–351.

Lahti, P. and Pulmannová, S. (2001). Coexistent vs. functional coexistence of quantum observables. *Reports on Mathematical Physics* **47**, 199–212.

Lahti, P., Pulmannová, S., and Ylinen, K. (1998). Coexistent observables and effects in convexity approach. *Journal of Mathematical Physics* **39**, 6364–6371.

Ludwig, G. (1964). Versuch einer axiomatischen Grundlegung der Quantenmechanik und allgemeinerer physikalischer Theorien, *Zeitschrift für Physik* **181**, 233–260.

Ludwig, G. (1967). Attempt of an Axiomatic Foundation of Quantum Mechanics and More General Theories. II, *Communications in Mathematical Physics* **4**, 331–348.

Ludwig, G. (1983). *Foundations of Quantum Mechanics*, Springer-Verlag, Berlin.

Mittelstaedt, P. (1978). *Quantum Logic*, Reidel, Dordrecht, The Netherlands.

Neumann, H. (1970). Seminar notes: coexistent effects and observables, *International School of Physics "Enrico Fermi,"* **IL** 407–411.

Ozawa, M. (1984). Quantum measuring processes of continuous observables. *Journal of Mathematical Physics* **25**, 79–87.

Pulmannová, S. (1980). Relative compatibility and joint distributions of observables, *Foundations of Physics* **10**, 641–653.

Varadarajan, V. S. (1962). Probability in physics and a theorem on simultaneous observability, *Communications in Pure and Applied Mathematics* **15**, 189–217.

Varadarajan, V. S. (1985). *Geometry of Quantum Theory*, Springer-Verlag, Berlin. First edition (in two volumes) by van Nostrand, Princeton, 1968, 1970.

von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton UP, Springer, Berlin. (Original German text, 1932)

Ylinen, K. (1985). On a theorem of Gudder on joint distributions of observables. In *Symposium on the Foundations of Modern Physics*, P. Lahti and P. Mittelstaedt, eds., World Scientific, Singapore, pp. 691–694.